Nonlocal Variational Problems
Problèmes variationnels non locaux
(Org: **Rustum Choksi** (McGill University) and/et **Ihsan Topaloglu** (Virginia Commonwealth University))

ANDREW BERNOFF, Harvey Mudd College
Biological Aggregation Driven by Social/Environmental Factors: A Nonlocal Model and Its Degenerate Cahn-Hilliard Approximation

Biological aggregations such as insect swarms and fish schools may arise from a combination of social interactions and environmental cues. Nonlocal continuum equations are often used to model aggregations, which manifest as localized solutions. While popular in the literature, the nonlocal models pose significant analytical and computational challenges. Beginning with the nonlocal aggregation model of [Topaz, Bertozzi & Lewis, Bull. Math. Bio., 2006], we derive the minimal well-posed long-wave approximation, which is a degenerate Cahn-Hilliard equation. Using analysis and computation, we study energy minimizers and show that they retain many salient features of those of the nonlocal model. Furthermore, using the Cahn-Hilliard model as a testbed, we investigate how an external potential modeling food sources can suppress peak population density, which is essential for controlling locust outbreaks. Random potentials tend to increase peak density, whereas periodic potentials can suppress it.

ALMUT BURCHARD, University of Toronto
On a non-local shape optimization problem related with swarming

I will describe recent work with Rustum Choksi and Ihsan Topaloglu on a shape optimization problem where the energy functional is given by an attractive/repulsive interaction potential in power-law form. A natural conjecture is that balls minimize this energy for large mass, and minimizers fail to exist if the mass falls below a certain critical threshold. We have partial results that support this view. Time permitting, I will discuss recent progress due to Frank and Lieb, and open questions.

RICCARDO CRISTOFERI, Carnegie Mellon University
Periodic critical points of the Ohta-Kawasaki functional

In this talk we present some new observations about periodic critical points and local minimizers of a nonlocal isoperimetric problem arising in the modeling of diblock copolymers. In particular, by using a purely variational procedure, we show that it is possible to construct (locally minimizing) periodic critical points whose shape resemble that of any given strictly stable constant mean curvature (periodic) hypersurface.

NESTOR GUILLEN, University of Massachusetts Amherst
Min-max formulas for nonlocal elliptic operators and applications

A mapping F between spaces of real valued functions is said to have the Global Comparison Property (GCP) if $u \leq v$ everywhere with $u = v$ at some point x means that $F(u) \leq F(v)$ at this point x. A classical result of Courrège says that a continuous linear map from $C^2(\mathbb{R}^d)$ to $C^0(\mathbb{R}^d)$ has the GCP if and only if it is a sum of jump and drift-diffusion operators. In work with Russell Schwab, we characterize nonlinear maps having the GCP as those given by a min-max of linear operators having the GCP. This result provides representation formulas for the Dirichlet-to-Neumann map of nonlinear elliptic equations, and for the interface velocity for various free boundary problems, respective applications will be discussed along with a list of related open questions.

VESA JULIN, University of Jyvaskyla
Stability of the Gaussian isoperimetric inequality

I will discuss about the stability of the Gaussian isoperimetric inequality and its connection with the Gaussian isoperimetric inequality for symmetric sets.

THEODORE KOLOKOLNIKOV, Dalhousie
The rise and fall of kings: globalization and class mobility

We introduce a nonlocal model of wealth distribution in a society that incorporates a spatial component of individual members. The resulting wealth distribution depends critically on the interaction distance. When this distance is small, a few individuals (which we call "kings") accrue most of the society's wealth and class mobility disappears: the kings persist in time. We call this regime the "feudal state". On the other hand, when the interaction is more global, the wealth of individuals fluctuates with time, preventing any single individual from dominating over long time periods, thus increasing class mobility. We show that there is a critical interaction distance (which we compute) at which the society "bifurcates" from the feudal state to a more socially mobile society.

XIN YANG LU, McGill University
Centroidal Voronoi tessellations and Gersho's conjecture in 3D

Centroidal Voronoi Tessellations (CVT) are tessellations using Voronoi regions of their centroids. CVTs are useful in data compression, optimal quadrature, optimal quantization, clustering, and optimal mesh generation. Many patterns seen in nature are closely approximated by a CVT. Examples of this include the Giant's Causeway, the cells of the cornea, and the breeding pits of the male tilapia.

This is closely related to Gersho's conjecture, which states that there exists an asymptotically optimal CVT whose Voronoi regions are all rescaled copies of the same polytope. Straightforward in 1D, and proven in 2D, Gersho's conjecture is still open for higher dimensions. One of the main difficulties is that Gersho's conjecture is a strongly nonlocal, infinite dimensional minimization problem (even in 3D). In this talk we will present some recent results which reduce Gersho's conjecture to a local, finite dimensional problem in 3D. Joint work with Rustum Choksi.

TADELE MENGESHA, University of Tennessee Knoxville
Calderon-Zygmund theory for the spectral fractional elliptic equations

Global Calderon-Zygmund type estimates are obtained for solutions to fractional elliptic problems over smooth domains. Our approach is based on the "extension problem" where the fractional elliptic operator is realized as a Dirichlet-to-Neumann map to a degenerate elliptic PDE in one more dimension. This allows the possibility of deriving estimates for solutions to the fractional elliptic equations from that of degenerate elliptic equations. We will confirm this first by obtaining weighted estimates for the gradient of solutions to a class of linear degenerate/singular elliptic problems over a bounded, possibly non-smooth, domain. The class consists of those with coefficient matrix that symmetric, nonnegative definite, and both its smallest and largest eigenvalues are proportion to a particular weight that belongs to a Muckenhoupt class. The weighted estimates are obtained under a smallness condition on the mean oscillation of the coefficients with a weight. This is a joint work with T. Phan.

MASSIMILIANO MORINI, University of Parma
Nonlinear stability results for the nonlocal Mullins-Sekerka flow.

It has been recently shown that strictly stable critical configurations for the sharp interface Ohta-Kawasaki energy are in fact isolated local minimizers with respect to small L^1-perturbations. After reviewing such results, we study the corresponding evolution problem and we show that such strictly stable configurations are exponentially stable for the $H^{-1/2}$-gradient flow of the Ohta-Kawasaki energy, also known as the nonlocal Mullins-Sekerka flow.

CYRILL MURATOV, NJIT

A universal thin film model for Ginzburg-Landau energy with dipolar interaction

We present an analytical treatment of a three-dimensional variational model of a system that exhibits a second-order phase transition in the presence of dipolar interac- tions. Within the framework of Ginzburg-Landau theory, we concentrate on the case in which the domain occupied by the sample has the shape of a flat thin film and obtain a reduced two-dimensional, non-local variational model that describes the ener- getics of the system in terms of the order parameter averages across the film thickness. Namely, we show that the reduced two-dimensional model is in a certain sense asymp- totically equivalent to the original three-dimensional model for small film thicknesses. Using this asymptotic equivalence, we analyze two different thin film limits for the full three-dimensional model via the methods of Γ-convergence applied to the reduced two- dimensional model. In the first regime, in which the film thickness vanishes while all other parameters remain fixed, we recover the local two-dimensional Ginzburg-Landau model. On the other hand, when the film thickness vanishes while the sample's lateral dimensions diverge at the right rate, we show that the system exhibits a transition from homogeneous to spatially modulated global energy minimizers. We identify a sharp threshold for this transition.

ROBIN NEUMAYER, University of Texas at Austin

Higher regularity of the free boundary for the fractional obstacle problem

In the obstacle problem for the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, Caffarelli, Salsa, and Silvestre showed that the free boundary is $C^{1,\alpha}$ near regular points. In this talk, based on joint work with Y. Jhaveri, we show that the free boundary is smooth at regular points provided the obstacle is smooth. The main idea of the proof is to establish a so-called higher order boundary Harnack principle as recently developed by De Silva and Savin.

DAVID SHIROKOFF, New Jersey Institute of Technology

Conic programming of a variational inequality for self-assembly

In this talk we examine a class of non-local, non-convex functionals that describe pairwise interactions in systems with a large number of particles. Although finding and verifying local minimizers to these energies is relatively straight-forward, the computation and verification of global minimizers is much more difficult, Here the global minimum is important as it is the most likely observable state in the presence of low thermal noise. We discuss how minimizing the energy functional can be viewed as testing whether an associated bilinear form is co-positive. We then examine sufficient conditions for global optimality which are obtained through a convex relaxation of the cone of co-positive functionals. The sufficient conditions are (i) sometimes sharp (for instance, in the case of a lattice minimizer), (ii) provide an optimal decomposition of the original energy functional into a convex/non-negative energy decomposition, and (iii) can also be used as a heuristic (with rigorous lower bounds) to compute the emerging shapes of self-assembled structures. It is conjectured that the solutions to the heuristic equations are also, in some cases, globally optimal.

WEIRAN SUN, Simon Fraser University

Radiative Transfer Equation with the Henyey-Greenstein Kernel

Radiative transfer equations with the Henyey-Greenstein kernel are often used to model light scattering in media such as animal tissues. In such models the forward-peakness of the scattering kernel is measured by an anisotropic factor g. It is known in the physics literature that asymptotic behaviour when $g \to 1$ is not the classical Fokker-Planck operator. Indeed in this talk we show that the limit should be a fractional Laplace operator on the sphere. Based on this analytical result, we design numerical schemes for approximating the scattering operator with the Henyey-Greenstein kernel. Unlike previous results when the mesh size depends on g and have to be refined as g approaches 1, our method is uniform in g. This reduces the computational cost when g is close to 1 and can provide an efficient scheme for solving RTE over the region where g varies in different parts.