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A counterexample to the Poincare-Birkhoff-Witt theorem

I will give a counterexample to the Poincare-Birkhoff-Witt theorem in characteristic p (but don't get too excited - in a tensor category that has no realization in vector spaces). Namely, let Ver_p be the Verlinde category over a field k of characteristic $p \geq 5$ (the semisimplification of $\mathrm{Rep}(\mathbb{Z}/p\mathbb{Z}, k)$), and let V be the object of Ver_p corresponding to the 2-dimensional indecomposable representation of $Z/p\mathbb{Z}$. Let $\mathfrak{g} = \mathrm{FLie}_{\leq p}(V)$ be the free Lie algebra of V truncated above degree p. Then $\mathfrak{g}$ fails the PBW theorem; in fact, it is not a Lie subalgebra of any associative algebra. To correct this, we need to add a new axiom to the usual Lie operad axioms in the definition of a Lie algebra. This is already familiar for Lie algebras in characteristic 2, where we need to add the condition that $[x,x]=0$, and for Lie superalgebras in characteristic 3, where we need to add the condition that $[[x,x],x]=0$ for odd x. Likewise, in characteristic $p \geq 5$ we need a new axiom which is homogeneous of degree p in a single entry x; I call it the p-Jacobi identity. I will write down this axiom and sketch a proof that together with the Lie operad axioms it suffices for the PBW theorem in Ver_p (I expect that it is so in any symmetric tensor category). For this, I will develop a theory of Koszul duality in symmetric tensor categories.