For a \( k \)-uniform hypergraph \( F \) let \( \text{ex}(n, F) \) be the maximum number of edges of a \( k \)-uniform \( n \)-vertex hypergraph \( H \) which contains no copy of \( F \). Determining or estimating \( \text{ex}(n, F) \) is a classical and central problem in extremal combinatorics. While for \( k = 2 \) this problem is well understood, due to the work of Turán and of Erdős and Stone, only very little is known for \( k \)-uniform hypergraphs for \( k > 2 \). We focus on the case when \( F \) is a \( k \)-uniform hypergraph with three edges on \( k + 1 \) vertices. Already this very innocent (and maybe somewhat particular looking) problem is still wide open even for \( k = 3 \).

We consider a variant of the problem where the large hypergraph \( H \) enjoys additional hereditary density conditions. Questions of this type were suggested by Erdős and Sós about 30 years ago. We show that every \( k \)-uniform hypergraph \( H \) with density \( > 2^{1-k} \) with respect to every large collection of \( k \)-cliques induced by sets of \( (k-2) \)-tuples contains a copy of \( F \). The required density \( 2^{1-k} \) is best possible as higher order tournament constructions show.

Our result can be viewed as a common generalisation of the first extremal result in graph theory due to Mantel (when \( k = 2 \) and the hereditary density condition reduces to a normal density condition) and a recent result of Glebov, Král’, and Volec (when \( k = 3 \) and large subsets of vertices of \( H \) induce a subhypergraph of density \( > 1/4 \)). Our proof for arbitrary \( k \geq 2 \) utilises the regularity method for hypergraphs.